Grade 6 Math Circles

March 26-28, 2024

Measurement & Number Systems - Problem Set

1. Which measurement unit (inches, feet, yards or miles) would be most appropriate to measure the following lengths?
   (a) The length of your hand
   (b) The distance between Toronto and Paris
   (c) The length of a football field

   Solution:
   (a) Using inches would be most appropriate.
   (b) Using miles would be most appropriate.
   (c) Using either feet or yards would be most appropriate.

2. Which measurement unit (teaspoons, fluid ounces, cups or gallons) would be most appropriate to measure the following volumes?
   (a) The volume of water required to fill a pool
   (b) The exact amount of vanilla to add to a pancake recipe
   (c) The amount of liquid in a water bottle

   Solution:
   (a) Using gallons would be most appropriate.
   (b) Using teaspoons would be most appropriate.
   (c) Using fluid ounces would be most appropriate.

3. A team of engineers from all around the world are working hard to build a spaceship. They have lots of great ideas, but they cannot decide which measurement system to use: US customary or metric. What system might you suggest? Why?

Solution: Because the engineers are from 'all around the world', it is probably most suitable to use metric measurement. US customary is a system unique to the United States, so using US customary units would make communicating their ideas unnecessarily difficult. If all of the engineers and astronauts were from the United States, then US customary would be appropriate. However, in this case, using metric is more sensible.

4. Over time, the exchange rate between the Canadian dollar (CAD) and US dollar (USD) has fluctuated. 50 years ago, in 1974, \$1.00 CAD was approximately \$1.01 USD. Today, a \$30.00 CAD calculator would cost \$22.09 USD.

   (a) What is the conversion factor from CAD to USD for each year?

   (b) How much would a \$50 CAD bill be worth in USD for each year?

Solution:

(a) The conversion factor in 1974 is simply $\dfrac{\$1.01 \text{ USD}}{\$1.00 \text{ CAD}}$. The conversion factor in 2024 is $\dfrac{\$22.09 \text{ USD}}{\$30.00 \text{ CAD}}$ or $\dfrac{\$1.00 \text{ USD}}{\$1.36 \text{ CAD}}$.

(b) In 1974, we have

$$\$50.00 \text{ CAD} = \$50.00\ \cancel{\text{CAD}} \times \frac{\$1.01 \text{ USD}}{\$1.00\ \cancel{\text{CAD}}} = \frac{\$50.50 \text{ USD}}{\$1.00} = \$50.50 \text{ USD}$$

In 2024, we have

$$\$50.00 \text{ CAD} = \$50.00\ \cancel{\text{CAD}} \times \frac{\$1.00 \text{ USD}}{\$1.36\ \cancel{\text{CAD}}} = \frac{\$50.00 \text{ USD}}{\$1.36} = \$36.76 \text{ USD}$$

5. The Saltwater Crocodile has one of the most powerful bites of any animal. The strongest recorded bite was 3,700 pounds of force per square inch (pound-force/in^2, or 'PSI'). How many Pascals (Pa) was this bite? Why is PSI a more suitable measurement?

Solution: From the conversion chart in the lesson, we see 1 PSI = 6894.76 Pa. We're told the bite of the crocodile has a pressure of 3,700 PSI, and we want to eliminate the units of PSI. This means we can write the conversion factor as $\dfrac{6894.76 \text{ Pa}}{1 \text{ PSI}}$. Multiplying this conversion factor with our starting value of 3,700 PSI, we get

$$3,700\ \cancel{\text{PSI}} \times \frac{6894.76\ \text{Pa}}{1\ \cancel{\text{PSI}}} = 3,700 \times 6894.76\ \text{Pa} = 25,510,612\ \text{Pa}$$

This is an enormous number to describe the strength of the bite, where 3,700 PSI is a much smaller number to work with. Typically, we try to avoid working with numbers in the millions which is why PSI is often used to describe the pressure an animal bites with. We could alternatively stick with units of Pascals, but instead of writing the entire number 25,510,612 we could simply write 25.51 MPa, where the 'M' stands for the prefix 'Mega' meaning 'million'.

6. The way we measure temperature varies around the world. Suppose the temperature in Celsius is C. The temperature in Fahrenheit, F, is calculated using the formula:

$$F = (1.8 \times C) + 32$$

(a) If Anu (who lives in Canada) measures the temperature to be 25°C, what temperature in Fahrenheit does Rachel (who lives in the United States) measure?

(b) Explain why this question can't be solved using conversion factors.

Solution:

(a) If Anu measures the temperature is 25° Celsius, we set $C = 25$. Using the formula, we see

$$F = (1.8 \times C) + 32 = F = (1.8 \times 25) + 32 = 45 + 32 = 77$$

Therefore, Rachel measures a temperature of $77°F$.

(b) The question cannot be solved using conversion factors because we cannot start with a temperature in Celsius and end up with a temperature in Fahrenheit by multiplying by a single value. We have to multiply by 1.8, *then* add 32. Because of the '+32' piece of the formula, there is no single conversion factor to switch from the two units — we must do two operations instead of one multiplication..

7. Convert each of the following binary numbers into decimal.

(a) 101 (b) 10010 (c) 11011101

Solution:

(a) Using the binary to decimal conversion process from the lesson, we have the binary number 101 is equal to

$$(1 \times 2^2) + \cancel{(0 \times 2^1)} + (1 \times 2^0) = (1 \times 4)) + (1 \times 1) = 4 + 0 + 1 = 5$$

(b) In decimal form, the binary number 10010 is equivalent to

$$(1 \times 2^4) + \cancel{(0 \times 2^3)} + \cancel{(0 \times 2^2)} + (1 \times 2^1) + \cancel{(0 \times 2^0)}$$

$$= (1 \times 16) + (1 \times 2) = 16 + 2 = 18$$

11011101 is equivalent to

$$(1 \times 2^7) + (1 \times 2^6) + \cancel{(0 \times 2^5)} + (1 \times 2^4) + (1 \times 2^3) + (1 \times 2^2) + \cancel{(0 \times 2^1)} + (1 \times 2^0)$$

$$= (1 \times 128) + (1 \times 64) + (1 \times 16) + (1 \times 8) + (1 \times 4) + (1 \times 1)$$

$$= 128 + 64 + 16 + 8 + 4 + 1 = 221$$

8. When handling data and information, we say 1 byte is equivalent to 8 bits. This is where the terms 'megabytes', 'gigabytes', and so on come from.

(a) How many bytes is a 32-digit binary number?

(b) What is the largest possible binary number we can make with this number of bytes? What is its decimal representation?

Solution:

(a) Each binary digit in a binary number is one bit. Therefore, a number with 32 binary digits has 32 bits. The number of bytes can be found using the conversion factor:

$$32 \cancel{\text{ bits}} \times \frac{1 \text{ byte}}{8 \cancel{\text{ bits}}} = \frac{32 \text{ bytes}}{4} = 8 \text{ bytes}$$

(b) As mentioned in the lesson, the largest possible binary number we can make with a certain number of digits has bit values of 1 for every digit. For 32 bits (or 8 bytes) the number would be 32 repeated 1-digits, as below:

$$11111111111111111111111111111111$$

Its decimal representation is found by using the remark from Example 3 of the lesson:

$$\boxed{\text{For an } n\text{-bit number, its largest decimal value is } 2^n - 1.}$$

Here, we have a 32-bit number, so its decimal representation is

$$2^n - 1 = 2^{32} - 1 = 4,294,967,295$$

Alternatively, we could perform the calculation below and get the same result.

$$2^{31} + 2^{30} + 2^{29} + ... + 2^2 + 2^1 + 2^0 = 4,294,967,295$$

9. Harry plans on travelling to Mexico for the summer. According to his research, the hotel will cost 950 Mexican pesos per night, his food will cost 200 pesos per day, his rental vehicle will cost 520 pesos per day, and all other expenses will cost about 350 pesos. If he plans to stay in Mexico for two weeks, how much money should Harry plan to spend in Canadian dollars? *Note:* \$1.00 CAD is equal to 12.37 Mexican pesos.

Solution: Based on Harry's research, the cost of each day will be

$$\frac{950 \text{ pesos}}{1 \text{ day}} + \frac{200 \text{ pesos}}{1 \text{ day}} + \frac{520 \text{ pesos}}{1 \text{ day}} + \frac{350 \text{ pesos}}{1 \text{ day}} = \frac{2,020 \text{ pesos}}{1 \text{ day}}$$

We then multiply by the conversion factor $\frac{7 \text{ days}}{1 \text{ week}}$:

$$\frac{2,020 \text{ pesos}}{1 \cancel{\text{day}}} \times \frac{7 \cancel{\text{days}}}{1 \text{ week}} = \frac{14,140 \text{ pesos}}{1 \text{ week}}$$

Next, we multiply by his total stay of 2 weeks:

$$2 \cancel{\text{weeks}} \times \frac{14,140 \text{ pesos}}{1 \cancel{\text{week}}} = 28,280 \text{ pesos}$$

Finally, we convert from pesos to Canadian dollars (CAD) using the conversion factor $\frac{\$1 \text{ CAD}}{12.37 \text{ pesos}}$:

$$28,280 \cancel{\text{pesos}} \times \frac{\$1 \text{ CAD}}{12.37 \cancel{\text{pesos}}} \simeq \$2,286 \text{ CAD}$$

Therefore Harry should plan to spend about \$2,300 CAD in the 2 weeks he is on vacation.

10. * A farmer hires 12 workers to pick apples in an orchard. Each group of 3 apples weighs one pound, and each worker can pick 23 pounds of apples per hour and work an 8-hour shift. If each apple is sold for 50¢, how much money (in dollars) does the farm make every day if it is open for 16 hours a day?

Solution: We are told that each worker can pick 23 pounds of apples per hour, so in 8 hours, each worker can pick

$$\frac{23 \text{ pounds}}{1 \cancel{\text{hour}}} \times 8 \cancel{\text{hours}} = 23 \text{ pounds} \times 8 = 184 \text{ pounds per worker per shift}$$

Or, in other words, $\frac{184 \text{ pounds}}{1 \text{ worker} \times 1 \text{ shift}}$. We also mustn't forget that the farm is open for 16 hours a day (or two 8-hour shifts for each worker). We then multiply our previous quantity by '2 shifts' to get

$$2 \cancel{\text{shifts}} \times \frac{184 \text{ pounds}}{1 \text{ worker} \times 1 \cancel{\text{shift}}} = \frac{368 \text{ pounds}}{1 \text{ worker}}$$

Next, we multiply by the number of workers (12):

$$\frac{368 \text{ pounds}}{1 \cancel{\text{worker}}} \times 12 \cancel{\text{workers}} = 4,416 \text{ pounds}$$

We are also told that 1 pound of apples is equivalent to a group of 3 apples. The conversion factor we get from this is $\frac{3 \text{ apples}}{1 \text{ pound}}$. Multiplying by this conversion factor, we cancel out the 'pounds' units:

$$4,416 \cancel{\text{pounds}} \times \frac{3 \text{ apples}}{1 \cancel{\text{pound}}} = 13,248 \text{ apples}$$

Each apple is worth 50¢ so we multiply by the conversion factor $\frac{50¢}{1 \text{ apple}}$:

$$13,248 \cancel{\text{apples}} \times \frac{50¢}{1 \cancel{\text{apple}}} = 662,400¢$$

Finally, we turn the 'cents' unit into 'dollars' by multiplying by the conversion factor $\frac{\$1}{100\cancel{¢}}$.

$$662,400\cancel{¢} \times \frac{\$1}{100\cancel{¢}} = \frac{\$662,400}{100} = \$6,624$$

Therefore, every day the farm makes \$6,624. We can also do this entire problem in a single line:

$$\frac{23 \cancel{\text{pounds}}}{1 \cancel{\text{worker}} \times 1 \cancel{\text{hour}}} \times \frac{8 \cancel{\text{hours}}}{1 \cancel{\text{shift}}} \times 12 \cancel{\text{workers}} \times \frac{3 \cancel{\text{apples}}}{1 \cancel{\text{pound}}} \times \frac{50\cancel{¢}}{1 \cancel{\text{apple}}} \times \frac{\$1}{100\cancel{¢}} = \$6,624$$